2022. 2. 10

- Recall the definition of Cesáro summability and Abel summability.
- $\sum_{n=1}^{\infty} c_n$ convergent $\Rightarrow$ Cesáro summable $\Rightarrow$ Abel summable

$\qquad\qquad\qquad\qquad\qquad\qquad \not\Leftarrow \qquad\qquad\qquad\qquad\qquad \not\Leftarrow$

Def (Cesáro summability)

Let $\sum_{n=1}^{\infty} c_n$ be a series of complex numbers.

Let $s_n := \sum_{k=1}^{n} c_k$ and $\sigma_N := \frac{1}{N}(s_1 + \cdots + s_N)$

We say $\sum_{n=1}^{\infty} c_n$ is Cesáro summable to s if $\sigma_N \to s \in \mathbb{C}$ as $N \to \infty$.

Def (Abel summability)

Let $\sum_{n=1}^{\infty} c_n$ be a series of complex numbers.

For any $0 \le r < 1$, set $A(r) := \sum_{n=1}^{\infty} c_n r^n$.

We say $\sum_{n=1}^{\infty} c_n$ is Abel summable to s if

(1) $A(r)$ converges for any $r \in [0, 1)$.

(2) $A(r) \to s \in \mathbb{C}$ as $r \to 1^-$.

- Convergence $\Rightarrow$ Cesáro summability.

Suppose $\sum_{n=1}^{\infty} c_n = s \in \mathbb{C}$ and $s_n := \sum_{k=1}^{n} c_k$

Then $s_n \to s$ as $n \to \infty$, i.e.

$\forall\, \varepsilon > 0,\ \exists\, N_0 \in \mathbb{N}$ s.t. $\forall\, n > N_0$ $|s_n - s| < \varepsilon$

Thus, $$|\sigma_N - s| = \left|\left(\frac{1}{N}\sum_{n=1}^{N} s_n\right) - s\right|$$

$$= \left|\frac{1}{N}\sum_{n=1}^{N}(s_n - s)\right|$$

△-ineq $$\leq \frac{1}{N}\sum_{n=1}^{N}|s_n - s|$$

$$= \frac{1}{N}\sum_{n=1}^{N_0}|s_n - s| + \frac{1}{N}\sum_{n=N_0+1}^{\infty}|s_n - s|$$

(Take $N_1 \in \mathbb{N}$ s.t. $N_1 > N_0$ and $\frac{1}{N_1} < \frac{\varepsilon}{M}$ $\left(\sum_{n=1}^{N_0}|s_n - s| < M\right)$)

$$< \frac{\varepsilon}{M}\cdot M + \varepsilon$$

$$= 2\varepsilon, \quad \forall\, N > N_1.$$

Therefore, we have $\sigma_N \to s$ as $N \to \infty$.

- Cesáro summability $\Rightarrow$ Abel summability.

Suppose $\sigma_N := \frac{1}{N}\sum_{n=1}^{N} s_n \to s$ as $N \to \infty$.

We want show (1) $A(r) := \sum_{n=1}^{\infty} c_n r^n$ converges, $\forall r \in [0,1)$;

(2) $A(r) \to s$ as $r \to 1^-$.

Notice that $c_1 = s_1 = \sigma_1$

$$c_2 = s_2 - s_1 = (2\sigma_2 - \sigma_1) - \sigma_1$$
$$= 2(\sigma_2 - \sigma_1).$$

When $n \geq 3$,

$$c_n = s_n - s_{n-1}$$
$$= (n\sigma_n - (n-1)\sigma_{n-1}) - ((n-1)\sigma_{n-1} - (n-2)\sigma_{n-2})$$
$$= n\sigma_n - 2(n-1)\sigma_{n-1} + (n-2)\sigma_{n-2}.$$

$$A(r) := \sum_{n=1}^{\infty} c_n r^n = \sigma_1 r + 2(\sigma_2 - \sigma_1) r^2 + \sum_{n=3}^{\infty}[n\sigma_n - 2(n-1)\sigma_{n-1} + (n-2)\sigma_{n-2}] r^n$$
$$= \sigma_1 r + 2(\sigma_2 - \sigma_1) r^2 + \sum_{n=3}^{\infty} n\sigma_n r^n - 2\sum_{n=2}^{\infty} n\sigma_n r^{n+1} + \sum_{n=1}^{N} n\sigma_n r^{n+2}$$
$$= \sum_{n=1}^{\infty} n\sigma_n r^n - 2\sum_{n=1}^{\infty} n\sigma_n r^{n+1} + \sum_{n=1}^{\infty} n\sigma_n r^{n+2}$$
$$= \sum_{n=1}^{\infty} n\sigma_n r^n (1 - r + r^2)$$
$$= (1-r)^2 \sum_{n=1}^{\infty} n\sigma_n r^n$$

$(|\sigma_n| < M)$ $\quad < M(1-r)^2 \sum_{n=1}^{\infty} n r^n$

$\left(\sum_{n=1}^{\infty} r^n = \frac{1}{1-r} \Downarrow \sum_{n=1}^{\infty} n r^n = \frac{r}{(1-r)^2} \right)$ $\quad = M(1-r)^2 \frac{r}{(1-r)^2}$, $\forall r \in [0,1)$

$$= Mr.$$

$$|A(r) - s| \leq |A(r) - sr| + |sr - s|$$

$$|A(r) - sr| = \left| (1-r)^2 \sum_{n=1}^{\infty} n\sigma_n r^n - s(1-r)^2 \sum_{n=1}^{\infty} n r^n \right|$$

$$= \left| (1-r)^2 \sum_{n=1}^{\infty} (\sigma_n - s) n r^n \right|$$

Recall $\sigma_n \to s$ as $n \to \infty$.

$\forall \varepsilon > 0$, $\exists N_0 \in \mathbb{N}$ s.t. $\forall n > N_0$, $|\sigma_n - s| < \varepsilon$.

So $\left| (1-r)^2 \sum_{n=1}^{\infty} (\sigma_n - s) n r^n \right| = \left| (1-r)^2 \sum_{n=1}^{N_0} (\sigma_n - s) n r^n \right|$

$$+ \left| (1-r)^2 \sum_{n=N_0+1}^{\infty} (\sigma_n - s) n r^n \right|.$$

$$\left| (1-r)^2 \sum_{n=N_0+1}^{\infty} (\sigma_n - s) n r^n \right| \leq (1-r)^2 \sum_{n=N_0+1}^{\infty} |\sigma_n - s| n r^n$$

$$< \varepsilon (1-r)^2 \sum_{n=N_0+1}^{\infty} n r^n$$

$$< \varepsilon (1-r)^2 \frac{r}{(1-r)^2}$$

$$= \varepsilon r < \varepsilon, \quad \forall r \in [0, 1).$$

$\exists M > 0$ depending on N_0 s.t.

$$\left| \sum_{n=N_0+1}^{\infty} (\sigma_n - s) n r^n \right| < M.$$

$\exists\, \delta > 0$ s.t $\forall\ r \in (1-\delta, 1)$

$$(1-r)^2 < \frac{\varepsilon}{M} \quad \text{and} \quad |sr - s| < \varepsilon.$$

Therefore, $|A(r) - s| \leq |A(r) - sr| + |sr - s|$

$$\leq \left|(1-r)^2 \sum_{n=1}^{N_0} (\sigma_n - s) n r^n\right|$$

$$+ \left|(1-r)^2 \sum_{n=N_0+1}^{\infty} (\sigma_n - s) n r^n\right|$$

$$+ |sr - s|$$

$$< \frac{\varepsilon}{M} \cdot M + \varepsilon + \varepsilon$$

$$= 3\varepsilon, \quad \forall\ r \in (1-\delta, 1).$$

Hence, $A(r) \to s$ as $r \to 1^-$.

- Abel summability $\not\Rightarrow$ Cesáro summability

We know $A(r) = (1-r)^2 \sum_{n=1}^{\infty} n \sigma_n r^n$

If $\sigma_n = \begin{cases} 0, & n \text{ odd} \\ 1, & n \text{ even} \end{cases}$,

then
$$
\begin{aligned}
A(r) &= (1-r)^2 \sum_{n=1}^{\infty} 2n r^{2n} \\
&= 2(1-r)^2 \sum_{n=1}^{\infty} n (r^2)^n \\
&= 2(1-r)^2 \frac{r^2}{(1-r^2)^2} \\
&= 2 \frac{r^2 (1-r)^2}{(1+r)^2 (1-r)^2} \to \frac{1}{2} \quad \text{as } r \to 1^-
\end{aligned}
$$

But σ_n does not converge.

Let $c_1 = \sigma_1 = 0$

$c_2 = 2(\sigma_2 - \sigma_1) = 2(1-0) = 2$

When $n \geq 3$,

$$
\begin{aligned}
c_n &= n\sigma_n - 2(n-1)\sigma_{n-1} + (n-2)\sigma_{n-2} \\
&= \begin{cases} 0 - 2(n-1) + 0 = -2(n-1), & n \text{ odd}, \\ n - 0 + (n-2) = 2(n-1), & n \text{ even}. \end{cases}
\end{aligned}
$$

You can check these c_n's give the σ_n's we want.

- Cesáro summability $\not\Rightarrow$ Convergence

$$\sigma_N = \frac{1}{N}(s_1 + \cdots + s_N)$$

If $s_n = \begin{cases} 0, & n \text{ odd} \\ 1, & n \text{ even} \end{cases}$,

then $\sigma_N \to \frac{1}{2}$ as $N \to \infty$.

But s_n does not converge.

To make $s_n = \begin{cases} 0, & n \text{ odd} \\ 1, & n \text{ even} \end{cases}$.

Let $c_1 = 0$

When $n \geq 2$

$$c_n = s_n - s_{n-1} = \begin{cases} -1, & n \text{ odd} \\ 1, & n \text{ even} \end{cases}.$$

$\square$

Q : Under which condition does Abel Summability imply convergence ?

(Tauberian Theorem)

If $nc_n \to 0$ as $n \to \infty$, then Abel summability implies convergence.

Pf: Suppose $A(r) := \sum_{n=1}^{\infty} c_n r^n$ converges, $\forall r \in [0, r)$,

$A(r) \to s$ as $r \to 1^-$

$nc_n \to 0$ as $n \to \infty$.

Then $\left| \sum_{n=1}^{N} c_n - s \right|$

$$= \left| \sum_{n=1}^{N} c_n - A(r) \right| + \left| A(r) - s \right|$$

$$= \left| \sum_{n=1}^{N} c_n - \sum_{n=1}^{N} c_n r^n \right| + \left| \sum_{n=N+1}^{\infty} c_n r^n \right| + \left| A(r) - s \right|$$

$$= \left| \sum_{n=1}^{N} c_n (1 - r^n) \right| + \left| \sum_{n=N+1}^{\infty} c_n r^n \right| + \left| A(r) - s \right|$$

$$\leq \left| \sum_{n=1}^{N} c_n n (1 - r) \right| + \left| \sum_{n=N+1}^{\infty} c_n n \frac{r^n}{n} \right| + \left| A(r) - s \right|$$

$$= (1-r) \sum_{n=1}^{N} |c_n| n + \frac{1}{N} \sum_{N+1}^{\infty} |c_n| n r^n + \left| A(r) - s \right|$$

Let $r = 1 - \frac{1}{N}$

And $\forall \varepsilon > 0$

$\exists\ N_0 \in \mathbb{N}$ s.t.

$\forall\ N > N_0$

$|c_n| n < \varepsilon$

$|A(r) - s| < \varepsilon$

$$\leq 2\varepsilon + \varepsilon \frac{1}{N} \frac{(1-\frac{1}{N})^N}{1-(1-\frac{1}{N})}$$

$$= \varepsilon \left(2 + (1-\frac{1}{N})^N \right)$$

$$< \varepsilon (2 + M)$$